

A Matricial Algorithm for Polynomial Refinement

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Abstract

In order to have a multiresolution analysis, the scaling function must be refinable. That is, it must be the linear combination of 2-dilation, \mathbb{Z} -translates of itself. Refinable functions used in connection with wavelets are typically compactly supported. In 2002, David Larson posed the question in his REU site, “Are all polynomials (of a single variable) finitely refinable?” That summer the author proved that the answer indeed was true using basic linear algebra. The result was presented in a number of talks but had not been typed up until now. The purpose of this short note is to record that particular proof.

Polynomial Refinement

A scaling function for a multiresolution analysis must be *2-refinable*, but not all refinable functions can be a scaling function.

Definition 1. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is *(2-)refinable* if there exists a sequence $\alpha \in \ell^\infty(\mathbb{Z})$ such that

$$f(\cdot) = \sum_{\ell \in \mathbb{Z}} \alpha_\ell f(2 \cdot - \ell). \quad (1)$$

If α is finitely supported, we say that f is *finitely (2-)refinable*. When 2 is replaced by $a \neq 0$ in (1), we say that f is *a-refinable*.

The focus on $a = 2$ is due to the early connection of refinability with wavelet theory. In approximation theory and signal processing the α in (1) is called the *mask* or *low pass filter sequence*, respectively.

Theorem 2. Let $p : \mathbb{R} \rightarrow \mathbb{C}$ be a polynomial of degree n . Let $\{\ell_i\}_{i=0}^n$ be any set of $n+1$ distinct integers. For any $a \neq 0$, there exists $\alpha \in \ell^\infty(\mathbb{Z})$ supported in $\{\ell_i\}$ such that

$$p(\cdot) = \sum_{i=0}^n \alpha_{\ell_i} p(a \cdot - \ell_i). \quad (2)$$

The proof is at its core the same as the proof of Theorem 1.8 in [1], which was independently discovered by Gustafson, Savir, and Spears a few years after the author of this note. However, they present it in a slightly different light in their paper and focus on the end result of refinability rather than the fact that for any polynomial any sequence of $n+1$ distinct shifts is associated with a refinement mask. Note that we will index our matrices and vectors starting with 0 to make the notation easier.

Proof. Since the polynomial is of degree n , it may be written in the form $p(x) = \sum_{k=0}^n p_k x^k$ with $p_n \neq 0$. Plugging this into (2), we obtain

$$\begin{aligned} p(x) &= \sum_{i=0}^n \alpha_{\ell_i} \sum_{k=0}^n p_k (ax - \ell_i)^k \\ &= \sum_{i=0}^n \alpha_{\ell_i} \sum_{k=0}^n p_k \sum_{j=0}^k \binom{k}{j} a^j x^j (-\ell_i)^{k-j} \\ &= \sum_{j=0}^n x^j a^j \sum_{k=j}^n p_k \binom{k}{j} \sum_{i=0}^n \alpha_{\ell_i} (-\ell_i)^{k-j}. \end{aligned}$$

A comparison of the monomial coefficients followed by a substitution ($\bar{k} = n - k + j$) yields

$$\begin{aligned} p_j &= a^j \sum_{k=j}^n p_k \binom{k}{j} \sum_{i=0}^n \alpha_{\ell_i} (-\ell_i)^{k-j} \quad \text{for each } 0 \leq j \leq n \\ &= a^j \sum_{k=j}^n p_{n-k+j} \binom{n-k+j}{j} \sum_{i=0}^n \alpha_{\ell_i} (-\ell_i)^{n-k} \quad \text{for each } 0 \leq j \leq n. \end{aligned}$$

This may be rewritten as a matrix equation

$$p = D_a C V m, \tag{3}$$

where D_a is the diagonal matrix with $(D_a)_{i,i} = a^i$ for $0 \leq i \leq n$, C is the invertible upper triangular matrix

$$C = \left(\begin{array}{cc} p_{n-j+i} \binom{n-j+i}{j} & ; \quad i \leq j \\ 0 & ; \quad \text{else} \end{array} \right)_{0 \leq i, j \leq n},$$

V is the invertible (since the shifts ℓ_i are distinct) Vandermonde matrix

$$V = ((-\ell_j)^{n-i})_{0 \leq i, j \leq n},$$

and m is the refinement mask $m = (\alpha_{\ell_i})_{i=0}^n$. □

Corollary 3. *All polynomials $p : \mathbb{R} \rightarrow \mathbb{C}$ are finitely 2-refinable.*

Henning Thielemann also independently proved this fact using different methods in [2].

Much is known about Vandermonde matrices, in fact they admit a well-known LU -decomposition [3]. Thus (3) also provides a quick algorithm for determining refinement masks of polynomials.

References

- [1] Gustafson, Paul; Savir, Nathan; Spears, Ely (2006-11-14), "A Characterization of Refinable Rational Functions," *American Journal of Undergraduate Research*, **5**(3): 1120
- [2] Thielemann, Henning (2011-04-13). "Polynomial Functions are Refinable," accepted *International Journal of Wavelets, Multiresolution and Information Processing*.
- [3] Turner, L. Richard (1966), "Inverse of the Vandermonde Matrix with Applications," *NASA Technical Note*.